Problem Set: The Ramsey–Cass–Koopmans Model

Advanced Macroeconomics — Dr Lei Pan — Total: 100 Marks

Instructions. Answer all questions. Show all mathematical derivations clearly. Answers without derivation receive limited credit. Time is discrete, $t = 0, 1, 2, \ldots$. There is no population growth and no depreciation. Technology evolves as

$$A_{t+1} = (1+g)A_t, \qquad g > 0.$$

There are L identical infinitely lived households. Define

$$k_t \equiv \frac{K_t}{A_t L}, \qquad \widetilde{c}_t \equiv \frac{c_t}{A_t}.$$

Firms operate a CRS production function

$$Y_t = F(K_t, A_t L_t^D) = A_t L_t^D f(k_t),$$

where $f'(k) > 0$, $f''(k) < 0$, and $\lim_{k\to 0} f'(k) = +\infty$.

Question 1: Competitive Equilibrium and the Planner Problem [Total: 60 marks]

The representative household has lifetime utility

$$U(c_0, c_1, \ldots) = \sum_{t=0}^{\infty} \beta^t u(c_t), \qquad 0 < \beta < 1,$$

where $u'(c) > 0$, $u''(c) < 0$, and $\lim_{c\to 0} u'(c) = +\infty$. The household budget constraint is

$$c_t + s_t = A_t w_t + (1 + r_t)s_{t-1}, \qquad s_{-1} = K_0/L.$$

(a) Derive the firm's first-order conditions

$$r_t = f'(k_t), \qquad w_t = f(k_t) - f'(k_t)k_t,$$

and show why CRS and perfect competition imply zero profits.

(b) Derive the household Euler equation

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_{t+1}.$$

Then explain why the Inada condition gives an interior consumption path.

(c) Use market clearing,

$$L_t^D = L, \qquad K_{t+1} = Ls_t,$$

to derive the RCK transition equations:

$$\widetilde{c}_t = f(k_t) + k_t - (1+g)k_{t+1},$$
$$\frac{u'(A_t\widetilde{c}_t)}{\beta u'(A_{t+1}\widetilde{c}_{t+1})} = 1 + f'(k_{t+1}).$$

For CRRA utility,

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \qquad \theta > 0,$$

derive

$$\left(\frac{\widetilde{c}_{t+1}}{\widetilde{c}_t}\right)^{\theta} = \frac{\beta[1 + f'(k_{t+1})]}{(1+g)^{\theta}}.$$

(d) Write down the social planner's problem and derive its first-order condition. Show that the planner's equations coincide with the competitive-equilibrium transition equations. What does this imply about Pareto efficiency in the RCK model?

Question 2: Steady State, Golden Rule, and Saddle-Path Stability [Total: 40 marks]

Assume CRRA utility and Cobb–Douglas production:

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \qquad f(k) = k^{\alpha}, \qquad 0 < \alpha < 1.$$

(a) Derive the steady-state conditions

$$\widetilde{c}^* = f(k^*) - gk^*, \qquad \beta[1 + f'(k^*)] = (1+g)^{\theta}.$$

Solve for k^* under Cobb–Douglas production and interpret k^* as the modified Golden Rule capital stock.

(b) Derive the Golden Rule capital stock from

$$\widetilde{c} = f(k) - gk.$$

Show that

$$f'(k_{\mathrm{GR}}) = g, \qquad k_{\mathrm{GR}} = \left(\frac{\alpha}{g}\right)^{1/(1-\alpha)}.$$

Under the well-defined-utility restriction

$$\beta(1+g)^{1-\theta} < 1,$$

prove that $k^* < k_{\mathrm{GR}}$.

(c) For

$$\alpha = \frac{1}{3}, \qquad \beta = 0.96, \qquad \theta = 2, \qquad g = 0.02,$$

compute k^*, y^*, $\widetilde{c}^*$, r^*, k_{GR}, and $\widetilde{c}_{\mathrm{GR}}$. Compare k^* with k_{GR}.

(d) Rewrite the RCK system as a first-order dynamic system in $(k_t, \widetilde{c}_t)$. Derive the $\Delta k = 0$ and $\Delta \widetilde{c} = 0$ loci. Then linearise the system around $(k^*, \widetilde{c}^*)$ and show that the steady state is saddle-path stable. Explain the role of the transversality condition.

Detailed Solutions

Solution to Question 1 [60 marks]

Part (a) [12 marks]

The representative firm solves

$$\max_{K_t, L_t^D} F(K_t, A_t L_t^D) - r_t K_t - w_t A_t L_t^D.$$

Because production is constant returns to scale, define

$$k_t = \frac{K_t}{A_t L_t^D}, \qquad F(K_t, A_t L_t^D) = A_t L_t^D f(k_t).$$

Thus profit can be written as

$$\pi_t = A_t L_t^D \left[f(k_t) - r_t k_t - w_t \right].$$

The first-order condition with respect to capital gives

$$\frac{\partial F(K_t, A_t L_t^D)}{\partial K_t} = r_t.$$

Since

$$F(K_t, A_t L_t^D) = A_t L_t^D f(k_t), \qquad k_t = \frac{K_t}{A_t L_t^D},$$

we have

$$\frac{\partial F}{\partial K_t} = A_t L_t^D f'(k_t) \frac{1}{A_t L_t^D} = f'(k_t).$$

Therefore,

$$\boxed{r_t = f'(k_t).}$$

The first-order condition with respect to effective labour gives

$$\frac{\partial F(K_t, A_t L_t^D)}{\partial (A_t L_t^D)} = w_t.$$

Using Euler's theorem for CRS production,

$$F_K K_t + F_E L_t^E A_t = w_t A_t L_t^D + r_t K_t = Y_t,$$

where $L_t^E = A_t L_t^D$. Dividing by $A_t L_t^D$ gives

$$w_t + r_t k_t = f(k_t).$$

Using $r_t = f'(k_t)$,

$$\boxed{w_t = f(k_t) - f'(k_t) k_t.}$$

Substituting the factor prices into profits:

$$\pi_t = A_t L_t^D \left[f(k_t) - f'(k_t) k_t - (f(k_t) - f'(k_t) k_t) \right] = 0.$$

Thus,

$$\boxed{\pi_t = 0.}$$

Under CRS and perfect competition, factor payments exhaust output: capital is paid its marginal product and effective labour receives the residual marginal product.
Marking guide: firm problem, 3; derivation of r_t, 3; derivation of w_t, 3; zero-profit result and interpretation, 3.

Part (b) [12 marks]

The household chooses a sequence $\{s_t\}_{t=0}^{\infty}$ to maximise

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + s_t = A_t w_t + (1 + r_t) s_{t-1}.$$

Equivalently,

$$c_t = A_t w_t + (1 + r_t)s_{t-1} - s_t.$$

The saving choice s_t affects only c_t and c_{t+1}:

$$c_t = A_t w_t + (1 + r_t)s_{t-1} - s_t,$$

$$c_{t+1} = A_{t+1} w_{t+1} + (1 + r_{t+1})s_t - s_{t+1}.$$

Therefore,

$$\frac{\partial c_t}{\partial s_t} = -1, \qquad \frac{\partial c_{t+1}}{\partial s_t} = 1 + r_{t+1}.$$

The first-order condition with respect to s_t is

$$\beta^t u'(c_t)(-1) + \beta^{t+1} u'(c_{t+1})(1 + r_{t+1}) = 0.$$

Rearranging gives

$$u'(c_t) = \beta(1 + r_{t+1})u'(c_{t+1}).$$

Hence,

$$\boxed{\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_{t+1}.}$$

The Inada condition,

$$\lim_{c \to 0} u'(c) = +\infty,$$

rules out zero consumption. If $c_t = 0$, the marginal utility of current consumption is infinite, so it cannot be optimal to reduce current consumption to zero when resources are positive. Similarly, if $c_{t+1} = 0$, the marginal utility of future consumption is infinite. Thus the optimal path is interior:

$$c_t > 0 \qquad \text{for all } t.$$

Marking guide: household problem, 3; identification of how s_t affects c_t and c_{t+1}, 3; Euler equation, 4; Inada/interior explanation, 2.

Part (c) [20 marks]

Market clearing in the labour market gives

$$L_t^D = L.$$

Capital-market clearing gives

$$K_{t+1} = L s_t.$$

Therefore,

$$s_t = \frac{K_{t+1}}{L}.$$

Using

$$k_{t+1} = \frac{K_{t+1}}{A_{t+1} L},$$

we get

$$K_{t+1} = A_{t+1} L k_{t+1}.$$

Hence,

$$s_t = A_{t+1} k_{t+1}.$$

Since

$$A_{t+1} = (1 + g)A_t,$$

we obtain

$$\boxed{s_t = (1 + g)A_t k_{t+1}.}$$

The household budget constraint is

$$c_t + s_t = A_t w_t + (1 + r_t)s_{t-1}.$$

Using capital-market clearing one period earlier,

$$s_{t-1} = A_t k_t.$$

Substitute $s_t = (1 + g)A_t k_{t+1}$ and $s_{t-1} = A_t k_t$:

$$c_t + (1 + g)A_t k_{t+1} = A_t w_t + (1 + r_t)A_t k_t.$$

Divide by A_t:

$$\frac{c_t}{A_t} + (1+g)k_{t+1} = w_t + (1+r_t)k_t.$$

By definition,

$$\widetilde{c}_t \equiv \frac{c_t}{A_t}.$$

Using firm optimality,

$$r_t = f'(k_t), \qquad w_t = f(k_t) - f'(k_t)k_t.$$

Therefore,

$$\widetilde{c}_t + (1+g)k_{t+1} = f(k_t) - f'(k_t)k_t + [1+f'(k_t)]k_t.$$

The terms involving $f'(k_t)k_t$ cancel:

$$\widetilde{c}_t + (1+g)k_{t+1} = f(k_t) + k_t.$$

Thus,

$$\boxed{\widetilde{c}_t = f(k_t) + k_t - (1+g)k_{t+1}.}$$

The Euler equation is

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_{t+1}.$$

Since

$$c_t = A_t\widetilde{c}_t, \qquad c_{t+1} = A_{t+1}\widetilde{c}_{t+1}, \qquad r_{t+1} = f'(k_{t+1}),$$

we get

$$\boxed{\frac{u'(A_t\widetilde{c}_t)}{\beta u'(A_{t+1}\widetilde{c}_{t+1})} = 1 + f'(k_{t+1}).}$$

Now suppose

$$u(c) = \frac{c^{1-\theta}-1}{1-\theta}, \qquad \theta > 0.$$

Then

$$u'(c) = c^{-\theta}.$$

The Euler equation becomes

$$\frac{(A_t\widetilde{c}_t)^{-\theta}}{\beta(A_{t+1}\widetilde{c}_{t+1})^{-\theta}} = 1 + f'(k_{t+1}).$$

Equivalently,

$$\frac{1}{\beta}\left(\frac{A_{t+1}\widetilde{c}_{t+1}}{A_t\widetilde{c}_t}\right)^{\theta} = 1 + f'(k_{t+1}).$$

Since

$$\frac{A_{t+1}}{A_t} = 1+g,$$

we obtain

$$\frac{1}{\beta}\left[(1+g)\frac{\widetilde{c}_{t+1}}{\widetilde{c}_t}\right]^{\theta} = 1 + f'(k_{t+1}).$$

Hence,

$$\left(\frac{\widetilde{c}_{t+1}}{\widetilde{c}_t}\right)^{\theta} = \frac{\beta[1+f'(k_{t+1})]}{(1+g)^{\theta}}.$$

Thus,

$$\boxed{\left(\frac{\widetilde{c}_{t+1}}{\widetilde{c}_t}\right)^{\theta} = \frac{\beta[1+f'(k_{t+1})]}{(1+g)^{\theta}}.}$$

Marking guide: market-clearing derivation of s_t, 4; resource transition equation, 6; Euler transition equation, 5; CRRA transformation, 5.

Part (d) **[16 marks]**

The social planner chooses feasible allocations to maximise the representative household's lifetime utility:

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the aggregate resource constraint

$$F(K_t, A_tL) = C_t + (K_{t+1} - K_t).$$

Divide by A_tL:

$$f(k_t) = \frac{C_t}{A_tL} + \frac{K_{t+1}}{A_tL} - \frac{K_t}{A_tL}.$$

Since consumption per worker is $c_t = C_t/L$, we have

$$\frac{C_t}{A_tL} = \frac{c_t}{A_t} = \widetilde{c}_t.$$

Also,

$$\frac{K_{t+1}}{A_tL} = \frac{K_{t+1}}{A_{t+1}L}\frac{A_{t+1}}{A_t} = (1+g)k_{t+1},$$

and

$$\frac{K_t}{A_tL} = k_t.$$

Thus the resource constraint is

$$f(k_t) = \widetilde{c}_t + (1+g)k_{t+1} - k_t,$$

or

$$\boxed{\widetilde{c}_t = f(k_t) + k_t - (1+g)k_{t+1}.}$$

Because

$$c_t = A_t\widetilde{c}_t,$$

the planner's objective can be written as

$$\sum_{t=0}^{\infty} \beta^t u(A_t\widetilde{c}_t).$$

The variable k_{t+1} affects $\widetilde{c}_t$ and $\widetilde{c}_{t+1}$. From the resource constraint,

$$\frac{\partial \widetilde{c}_t}{\partial k_{t+1}} = -(1+g),$$

and

$$\widetilde{c}_{t+1} = f(k_{t+1}) + k_{t+1} - (1+g)k_{t+2},$$

so

$$\frac{\partial \widetilde{c}_{t+1}}{\partial k_{t+1}} = f'(k_{t+1}) + 1.$$

The first-order condition with respect to k_{t+1} is

$$\beta^t u'(A_t\widetilde{c}_t)A_t[-(1+g)] + \beta^{t+1}u'(A_{t+1}\widetilde{c}_{t+1})A_{t+1}[1 + f'(k_{t+1})] = 0.$$

Since

$$A_{t+1} = (1+g)A_t,$$

this becomes

$$-\beta^t u'(A_t\widetilde{c}_t)A_{t+1} + \beta^{t+1}u'(A_{t+1}\widetilde{c}_{t+1})A_{t+1}[1 + f'(k_{t+1})] = 0.$$

Cancel $\beta^t A_{t+1}$:

$$-u'(A_t\widetilde{c}_t) + \beta u'(A_{t+1}\widetilde{c}_{t+1})[1 + f'(k_{t+1})] = 0.$$

Hence,

$$\boxed{\frac{u'(A_t\widetilde{c}_t)}{\beta u'(A_{t+1}\widetilde{c}_{t+1})} = 1 + f'(k_{t+1}).}$$

Therefore, the planner's allocation is characterised by

$$\widetilde{c}_t = f(k_t) + k_t - (1+g)k_{t+1},$$

and

$$\frac{u'(A_t\widetilde{c}_t)}{\beta u'(A_{t+1}\widetilde{c}_{t+1})} = 1 + f'(k_{t+1}).$$

These are exactly the transition equations of the competitive equilibrium. Hence the competitive equilibrium coincides with the social planner's allocation. Therefore, under the maintained assumptions of perfect competition, complete markets, no externalities, and identical infinitely lived agents, the competitive equilibrium is Pareto-efficient.

Marking guide: planner problem, 4; resource constraint in intensive form, 4; planner FOC, 5; equivalence and Pareto-efficiency interpretation, 3.

Solution to Question 2 [40 marks]

Part (a) [10 marks]

From Question 1, the RCK transition equations under CRRA utility are

$$\widetilde{c}_t = f(k_t) + k_t - (1+g)k_{t+1},$$

and

$$\left(\frac{\widetilde{c}_{t+1}}{\widetilde{c}_t}\right)^\theta = \frac{\beta[1 + f'(k_{t+1})]}{(1+g)^\theta}.$$

A steady state satisfies

$$k_{t+1} = k_t = k^*, \qquad \widetilde{c}_{t+1} = \widetilde{c}_t = \widetilde{c}^*.$$

Substituting into the resource equation:

$$\widetilde{c}^* = f(k^*) + k^* - (1+g)k^*.$$

Therefore,

$$\boxed{\widetilde{c}^* = f(k^*) - gk^*.}$$

Substituting into the Euler equation:

$$1 = \frac{\beta[1 + f'(k^*)]}{(1+g)^\theta}.$$

Thus,

$$\boxed{\beta[1 + f'(k^*)] = (1+g)^\theta.}$$

Equivalently,

$$\boxed{f'(k^*) = \frac{(1+g)^\theta}{\beta} - 1.}$$

With Cobb–Douglas production,

$$f(k) = k^\alpha, \qquad f'(k) = \alpha k^{\alpha-1}.$$

Therefore,

$$\alpha(k^*)^{\alpha-1} = \frac{(1+g)^\theta}{\beta} - 1.$$

Since $\alpha - 1 < 0$, rewrite as

$$(k^*)^{1-\alpha} = \frac{\alpha}{\frac{(1+g)^\theta}{\beta} - 1}.$$

Hence,

$$\boxed{k^* = \left[\frac{\alpha}{\frac{(1+g)^\theta}{\beta} - 1}\right]^{1/(1-\alpha)}.}$$

This k^* is the modified Golden Rule capital stock. It is the capital stock that maximises the representative household's discounted lifetime utility, not the capital stock that maximises sustainable consumption per unit of effective labour.
Marking guide: steady-state resource condition, 3; steady-state Euler condition, 3; Cobb–Douglas solution for k^*, 3; interpretation as modified Golden Rule, 1.

Part (b) [10 marks]

The Golden Rule capital stock maximises steady-state consumption per unit of effective labour:

$$\widetilde{c} = f(k) - gk.$$

The Golden Rule problem is

$$\max_k \left[f(k) - gk\right].$$

The first-order condition is

$$f'(k_{\mathrm{GR}}) - g = 0.$$

Thus,

$$\boxed{f'(k_{\mathrm{GR}}) = g.}$$

For Cobb–Douglas production,

$$f'(k) = \alpha k^{\alpha-1}.$$

Hence,

$$\alpha k_{\text{GR}}^{\alpha-1} = g.$$

Rearrange:

$$k_{\text{GR}}^{1-\alpha} = \frac{\alpha}{g}.$$

Therefore,

$$\boxed{k_{\text{GR}} = \left(\frac{\alpha}{g}\right)^{1/(1-\alpha)}.}$$

Now compare k^* and k_{GR}. The modified Golden Rule satisfies

$$f'(k^*) = \frac{(1+g)^\theta}{\beta} - 1.$$

The Golden Rule satisfies

$$f'(k_{\text{GR}}) = g.$$

The restriction

$$\beta(1+g)^{1-\theta} < 1$$

implies

$$\beta < (1+g)^{\theta-1}.$$

Taking reciprocals,

$$\frac{1}{\beta} > (1+g)^{1-\theta}.$$

Multiplying by $(1+g)^\theta$,

$$\frac{(1+g)^\theta}{\beta} > 1+g.$$

Therefore,

$$\frac{(1+g)^\theta}{\beta} - 1 > g.$$

Thus,

$$f'(k^*) > f'(k_{\text{GR}}).$$

Since

$$f''(k) < 0,$$

the marginal product of capital is strictly decreasing in k. Therefore,

$$\boxed{k^* < k_{\text{GR}}.}$$

Hence, in the RCK model, the competitive equilibrium converges to the modified Golden Rule, which lies below the consumption-maximising Golden Rule capital stock under the well-defined-utility restriction.
Marking guide: Golden Rule objective, 2; FOC, 2; Cobb–Douglas k_{GR}, 2; proof that $k^* < k_{\text{GR}}$, 4.

Part (c) **[8 marks]**

The parameters are

$$\alpha = \frac{1}{3}, \qquad \beta = 0.96, \qquad \theta = 2, \qquad g = 0.02.$$

The steady-state Euler equation implies

$$r^* = f'(k^*) = \frac{(1+g)^\theta}{\beta} - 1.$$

Therefore,

$$r^* = \frac{(1.02)^2}{0.96} - 1 = \frac{1.0404}{0.96} - 1 = 0.08375.$$

Thus,

$$\boxed{r^* \approx 0.0838.}$$

Since

$$f'(k) = \alpha k^{\alpha-1},$$

we have

$$\frac{1}{3}(k^*)^{-2/3} = 0.08375.$$

Hence,

$$k^* = \left(\frac{1/3}{0.08375}\right)^{3/2} \approx 7.9404.$$

Thus,

$$\boxed{k^* \approx 7.9404.}$$

Output per unit of effective labour is

$$y^* = (k^*)^{1/3}.$$

Therefore,

$$\boxed{y^* \approx 1.9950.}$$

Steady-state consumption per unit of effective labour is

$$\widetilde{c}^* = f(k^*) - gk^*.$$

Thus,

$$\widetilde{c}^* = 1.9950 - 0.02(7.9404) \approx 1.8362.$$

Hence,

$$\boxed{\widetilde{c}^* \approx 1.8362.}$$

The Golden Rule capital stock satisfies

$$k_{\text{GR}} = \left(\frac{\alpha}{g}\right)^{1/(1-\alpha)} = \left(\frac{1/3}{0.02}\right)^{3/2}.$$

Thus,

$$\boxed{k_{\text{GR}} \approx 68.0414.}$$

Golden Rule consumption is

$$\widetilde{c}_{\text{GR}} = f(k_{\text{GR}}) - gk_{\text{GR}}.$$

Therefore,

$$\boxed{\widetilde{c}_{\text{GR}} \approx 2.7217.}$$

Comparison:

$$k^* \approx 7.9404 < 68.0414 \approx k_{\text{GR}}.$$

Therefore, the economy accumulates less capital than the Golden Rule level. This is not dynamic inefficiency; rather, it reflects optimal impatience. The competitive equilibrium is Pareto-efficient but does not maximise steady-state consumption because the representative household discounts the future.

Marking guide: r^*, 1; k^*, 2; y^* and $\widetilde{c}^*$, 2; k_{GR} and $\widetilde{c}_{\text{GR}}$, 2; comparison and interpretation, 1.

Part (d) [12 marks]

The resource transition equation is

$$\widetilde{c}_t = f(k_t) + k_t - (1+g)k_{t+1}.$$

Solving for k_{t+1} gives

$$\boxed{k_{t+1} = G(k_t, \widetilde{c}_t) = \frac{f(k_t) + k_t - \widetilde{c}_t}{1+g}.}$$

The Euler equation under CRRA utility is

$$\left(\frac{\widetilde{c}_{t+1}}{\widetilde{c}_t}\right)^\theta = \frac{\beta[1 + f'(k_{t+1})]}{(1+g)^\theta}.$$

Therefore,

$$\boxed{\widetilde{c}_{t+1} = H(k_t, \widetilde{c}_t) = \widetilde{c}_t \left[\frac{\beta[1 + f'(k_{t+1})]}{(1+g)^\theta}\right]^{1/\theta},}$$

where

$$k_{t+1} = G(k_t, \widetilde{c}_t).$$

The $\Delta k = 0$ locus is defined by

$$k_{t+1} = k_t.$$

Using the resource equation:

$$k_t = \frac{f(k_t) + k_t - \widetilde{c}_t}{1+g}.$$

Multiply by $1+g$:

$$(1+g)k_t = f(k_t) + k_t - \widetilde{c}_t.$$

Thus,

$$\boxed{\widetilde{c}_t = f(k_t) - gk_t.}$$

Capital rises when

$$k_{t+1} > k_t \iff \widetilde{c}_t < f(k_t) - gk_t.$$

Capital falls when

$$\widetilde{c}_t > f(k_t) - gk_t.$$

The $\Delta\widetilde{c} = 0$ condition is

$$\widetilde{c}_{t+1} = \widetilde{c}_t.$$

From the Euler equation:

$$1 = \left[\frac{\beta[1+f'(k_{t+1})]}{(1+g)^\theta}\right]^{1/\theta}.$$

Thus,

$$\beta[1+f'(k_{t+1})] = (1+g)^\theta.$$

This requires

$$k_{t+1} = k^*.$$

Therefore, using the resource transition equation, the $\Delta\widetilde{c} = 0$ locus in the $(k_t, \widetilde{c}_t)$ plane satisfies

$$\boxed{\widetilde{c}_t = f(k_t) + k_t - (1+g)k^*.}$$

Consumption rises when

$$k_{t+1} < k^*$$

because then

$$f'(k_{t+1}) > f'(k^*),$$

which implies

$$\frac{\beta[1+f'(k_{t+1})]}{(1+g)^\theta} > 1.$$

Consumption falls when

$$k_{t+1} > k^*.$$

Now linearise the system around the steady state. Define

$$\Phi(k) \equiv \left[\frac{\beta[1+f'(k)]}{(1+g)^\theta}\right]^{1/\theta}.$$

Then

$$k_{t+1} = G(k_t, \widetilde{c}_t) = \frac{f(k_t) + k_t - \widetilde{c}_t}{1+g},$$

and

$$\widetilde{c}_{t+1} = \widetilde{c}_t \Phi(k_{t+1}).$$

At the steady state,

$$\Phi(k^*) = 1.$$

Let deviations from steady state be

$$\widehat{k}_t = k_t - k^*, \qquad \widehat{c}_t = \widetilde{c}_t - \widetilde{c}^*.$$

The derivatives of G at the steady state are

$$G_k = \frac{1+f'(k^*)}{1+g} \equiv a, \qquad G_c = -\frac{1}{1+g} \equiv b.$$

Thus,

$$\widehat{k}_{t+1} = a\widehat{k}_t + b\widehat{c}_t.$$

Next,

$$\widetilde{c}_{t+1} = \widetilde{c}_t \Phi(k_{t+1}).$$

Linearising:

$$\widehat{c}_{t+1} = \widehat{c}_t + \widetilde{c}^* \Phi'(k^*) \widehat{k}_{t+1}.$$

Because

$$\Phi'(k^*) = \frac{f''(k^*)}{\theta[1 + f'(k^*)]},$$

define

$$\eta \equiv \Phi'(k^*) < 0.$$

Then

$$\widehat{c}_{t+1} = \widehat{c}_t + \widetilde{c}^* \eta (a\widehat{k}_t + b\widehat{c}_t).$$

Therefore, the linearised system is

$$\begin{pmatrix} \widehat{k}_{t+1} \\ \widehat{c}_{t+1} \end{pmatrix} = J \begin{pmatrix} \widehat{k}_t \\ \widehat{c}_t \end{pmatrix},$$

where

$$\boxed{J = \begin{pmatrix} a & b \\ \widetilde{c}^* \eta a & 1 + \widetilde{c}^* \eta b \end{pmatrix}.}$$

The determinant of J is

$$\det J = a(1 + \widetilde{c}^* \eta b) - b(\widetilde{c}^* \eta a) = a.$$

Thus,

$$\boxed{\det J = a = \frac{1 + f'(k^*)}{1 + g} > 0.}$$

The characteristic polynomial is

$$p(\lambda) = \lambda^2 - \operatorname{tr}(J)\lambda + \det J.$$

Evaluate it at $\lambda = 1$:

$$p(1) = 1 - \operatorname{tr}(J) + \det J.$$

Since

$$\operatorname{tr}(J) = a + 1 + \widetilde{c}^* \eta b, \qquad \det J = a,$$

we get

$$p(1) = 1 - (a + 1 + \widetilde{c}^* \eta b) + a = -\widetilde{c}^* \eta b.$$

Now

$$\widetilde{c}^* > 0, \qquad \eta < 0, \qquad b = -\frac{1}{1+g} < 0.$$

Hence,

$$\widetilde{c}^* \eta b > 0,$$

so

$$\boxed{p(1) < 0.}$$

Also,

$$p(0) = \det J = a > 0.$$

Therefore, since $p(0) > 0$ and $p(1) < 0$, one eigenvalue lies between 0 and 1. Since $p(1) < 0$ and $p(\lambda) \to +\infty$ as $\lambda \to +\infty$, the other eigenvalue is greater than 1. Hence the steady state has one stable eigenvalue and one unstable eigenvalue.

Therefore,

$$\boxed{(k^*, \widetilde{c}^*) \text{ is saddle-path stable.}}$$

Economic interpretation: k_0 is predetermined by history, but $\widetilde{c}_0$ is a forward-looking jump variable. Only one initial value of $\widetilde{c}_0$ places the economy on the stable saddle path. Other values of $\widetilde{c}_0$ generate paths that either violate feasibility or fail to converge. The transversality condition rules out these non-optimal explosive paths by requiring that the household not leave valuable capital unconsumed forever and not finance consumption through unsustainable asset positions.

Marking guide: first-order system, 3; $\Delta k = 0$ and $\Delta \widetilde{c} = 0$ loci, 3; linearised Jacobian, 3; saddle-path proof and TVC interpretation, 3.